

EXISTENCE AND MULTIPLICITY RESULTS FOR NONLINEAR CRITICAL NEUMANN PROBLEM ON COMPACT RIEMANNIAN MANIFOLDS.

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ABSTRACT. In this work, we study on compact Riemannian manifolds with boundary, the problems of existence and multiplicity of solutions to a Neumann problem involving the p -Laplacian operator and critical Sobolev exponents.

1. INTRODUCTION

Let (\overline{M}, g) be an $n(n \geq 3)$ -dimensional Riemannian manifolds with interior M and boundary ∂M which is an $(n - 1)$ -dimensional Riemannian manifold with induced metric g .

In this paper, we are interested in the problem of finding solutions on \overline{M} of the nonlinear Neumann boundary value problem

$$(1.1) \quad \begin{cases} \Delta_p u + a(x)|u|^{p^*-2}u = f(x)|u|^{p^*-2}u & \text{in } M; \\ |\nabla_g u|^{p-2}\partial_{\nu_g} u + k(x)|u|^{p^*-2}u = K(x)|u|^{p^*-2}u & \text{on } \partial M. \end{cases}$$

where ∂_{ν_g} is the outer unit normal derivative, $p \in (1, n)$, $\Delta_p = -\operatorname{div}(|\nabla_g u|^{p-2}\nabla_g u)$ is the p -Laplacian operator, $p^* = \frac{np}{n-p}$ and $p^{**} = \frac{(n-1)p}{n-p}$.

The case $p = 2$ corresponds to the famous problem of prescribing scalar and mean curvatures which has been studied by several authors, we cite for instance [21, 22], [14], [3], [4], [19].

For equation (1.1) without boundary condition, existence and multiplicity results on compact and complete manifolds are obtained by some authors [6], [8, 9, 10] and [20].

Problems of type of (1.1) are studied recently by a number of authors. For example, in [17], the authors proved an existence results on solid torus in \mathbb{R}^3 under the conditions that either the function f is positive and K is arbitrary or f is nonnegative and K is positive jointly with some condition on the function a . Successively, in [18] they obtained similar result on general Riemannian manifolds under the same conditions.

Independently, in [23], the authors dealt with problem (1.1) in the case where the function f changes sign and the function K is non positive. They obtained an existence result for the critical case and multiplicity result for the subcritical case. They used a fibering method introduced by Pohozaev[25].

In the present work, we are interested in the case where both functions f and K change sign on \overline{M} . Needless to say, this situation is more complicated and shows serious difficulties especially when we want to define a suitable constraint set.

In our study, we will adapt to our case some variational techniques introduced in

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[26] where the author dealt with a prescribed scalar curvature equation on compact manifolds without boundary. These techniques rely on considering minimization problem on suitable sets from which one can construct a sequence of curves satisfying a certain geometry that allows to get multiplicity of solutions in the subcritical case. By imposing additional conditions, we improve our results by proving multiplicity of solutions for problem (1.1) which includes the geometric case. Note that these techniques have been adapted for equation (1.1) without boundary condition in [10] and for a Q -curvature equation in [7].

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2. NOTATIONS

In the whole of the paper we denote by,

- (1) a and k two negative constants,
- (2) f and K two changing functions respectively on M and ∂M ,
- (3) q and r two constants such that $p < q \leq p^*, p < r \leq p^{**}$ and $r < q$,
- (4) $\|u\|_{p,M}, \|u\|_{p,\partial M}$ the L_p norms respectively on M and ∂M ,
- (5) $H_1^p(M)$ the Sobolev space of the functions in L_p with gradient in L_p ,
- (6) K_1, K_2 the best constants defined in the Sobolev and trace Sobolev inequalities which are the best constants such that there exist positive constants A and B such that

$$\begin{aligned} \|u\|_{q,M}^p &\leq K_1 \|\nabla_g u\|_{p,M}^p + A \|u\|_{p,M}^p, p < q \leq p^* \\ \|u\|_{r,\partial M}^p &\leq K_2 \|\nabla_g u\|_{p,M}^p + B \|u\|_{p,\partial M}^p, p < r \leq p^{**} \end{aligned}$$

- (7) $h^- = \min(h, 0), h^+ = \max(h, 0)$, respectively the negative and positive part of a function h .

By a solution of problem (1.1), we mean a function $u \in H_1^p(M)$ such that for every $v \in C^\infty(\overline{M})$ we have

$$\begin{aligned} \int_M |\nabla_g u|^{p-2} g(\nabla_g u, \nabla_g v) dv_g + \int_M a |u|^{p-2} u v dv_g + \int_{\partial M} k |u|^{p-2} u v d\sigma_g \\ = \int_M f |u|^{p^*-2} u v dv_g + \int_{\partial M} K |u|^{p^{**}-2} u v d\sigma_g. \end{aligned}$$

By regularity results [24], we get that $u \in C^{1,\beta}(\overline{M})$, for some $\beta \in (0, 1)$.

3. STATEMENT OF THE RESULTS

Our purpose is to prove existence and multiplicity of solutions to problem (1.1), it is to seek critical points $u \in H_1^p(M)$ of the energy functional

$$\begin{aligned} E(u) = \int_M |\nabla_g u|^p dv_g + a \int_M |u|^p dv_g + k \int_{\partial M} |u|^p d\sigma_g \\ - \frac{n-p}{n} \int_M f |u|^{p^*} dv_g - \frac{n-p}{n-1} \int_{\partial M} K |u|^{p^{**}} d\sigma_g. \end{aligned}$$

Define the quantity

$$\lambda_{f,K} = \inf_{\mathcal{A}} \frac{\|\nabla_g u\|_{p,M}^p}{|a| \|u\|_{p,M}^p + |k| \|u\|_{p,\partial M}^p}$$

where

$$\mathcal{A} = \{u \in H_1^p(M), u \geq 0 : \int_M |f^-| u dv_g + \int_{\partial M} |K^-| u d\sigma_g = 0\}$$

In this paper, we prove the following theorems

Theorem 3.1. *Let (M, g) be a compact Riemannian manifold with smooth boundary ∂M .*

There exist two positive constants N and H such that if the functions f and K satisfies the following conditions

- (1) $\lambda_{f,K} > 1$,
- (2) $\frac{\sup_M f}{\int_M |f^-| dv_g} < N, \sup_M f > 0$
- (3) $\frac{\sup_{\partial M} K}{\int_M |K^-| d\sigma_g} < H, \sup_{\partial M} K > 0$

then problem (1.1) admits a solution.

The following theorem concerns the multiplicity of the problem

$$(3.1) \quad \begin{cases} \Delta_p u + a(x)|u|^{p-2}u = f(x)|u|^{q-2}u & \text{in } M; \\ |\nabla_g u|^{p-2}\partial_{\nu_g} u + k(x)|u|^{p-2}u = K(x)|u|^{r-2}u & \text{on } \partial M. \end{cases}$$

where $p < q < p^*$ and $p < r < p^{**}$.

Theorem 3.2. *Let (M, g) be a compact Riemannian manifold with smooth boundary ∂M .*

There exist two positive constants N and H such that if the functions f and K satisfy the following conditions

- (1) $\lambda_{f,K} > 1$,
- (2) $\frac{\sup_M f}{\int_M |f^-| dv_g} < N, \sup_M f > 0$,
- (3) $\frac{\sup_{\partial M} K}{\int_M |K^-| d\sigma_g} < H, \sup_{\partial M} K > 0$,

then problem (3.1) admits at least two distinct solutions.

Under further conditions we prove the multiplicity of problem (1.1). Let u be the solution given by theorem 3.1 and N, H be the constants given in theorem 3.2. Denote I the functional

$$I(u) = \int_M |\nabla_g u|^p dv_g + a \int_M |u|^p dv_g + k \int_{\partial M} |u|^p d\sigma_g.$$

We prove the following theorem

Theorem 3.3. *Let (M, g) be a compact Riemannian manifold with smooth boundary ∂M .*

Suppose that the functions f and K satisfies the following conditions

- (1) $\lambda_{f,K} > 1$,
- (2) $\frac{\sup_M f}{\int_M |f^-| dv_g} < N, \sup_M f > 0$,
- (3) $\frac{\sup_{\partial M} K}{\int_M |K^-| d\sigma_g} < H, \sup_{\partial M} K > 0$,
- (4) $\frac{p(n-1)}{n(p-1)} (\sup_M f)^{-\frac{n-p}{p}} (\sup_{\partial M} K)^{\frac{n-p}{p-1}} K_1^{-n} K_2^{\frac{p(n-1)}{p-1}} \leq 1$

and that there exists a positive function $\Phi \in H_1^p(M)$ such that $I(\Phi) > 0$, $\int_M f \Phi^{p^} dv_g > 0$, $\int_{\partial M} K \Phi^{p^{**}} d\sigma_g > 0$ and*

$$0 < \sup_{\lambda \in [0,1]} E(\lambda \Phi) < E(u) + \frac{p}{n} \left[\sup_M f \right]^{1-\frac{n}{p}} K_1^{-\frac{n}{p}}$$

then, problem (1.1) admits at least two distinct solutions.

4. NECESSARY CONDITION

Recall that we have defined $\lambda_{f,K}$ as

$$\lambda_{f,K} = \inf_{\mathcal{A}} \frac{\|\nabla_g u\|_{p,M}^p}{|a|\|u\|_{p,M}^p + |k|\|u\|_{p,\partial M}^p}$$

where

$$\mathcal{A} = \{u \in H_1^p(M), u \geq 0 : \int_M |f^-| u dv_g + \int_{\partial M} |K^-| u d\sigma_g = 0\}$$

Let us prove the following lemma

Lemma 4.1. *If the problem 1.1 admits a solution, then $\lambda_{f,K} \geq 1$*

Proof. First, we show that $\lambda_{f,K}$ is attained. By homogeneity, we may take $\{u_i\} \subset H_1^p(M), u_i \geq 0$ a minimizing sequence for $\lambda_{f,K}$ such that $\|u_i\|_{p,M}^p + \|u_i\|_{p,\partial M}^p = 1$, then u_i is bounded in $H_1^p(M)$. By the Rellich-Kondrakov and Banach theorems, there exist a subsequence u_i and a function u such that u_i converges weakly in $H_1^p(M)$, strongly in $L_s(M)$ and $L_t(\partial M)$, $s < p^*, t < p^{**}$, almost everywhere in M and in the sens of trace on ∂M . Then we get $\|u\|_{p,M}^p + \|u\|_{p,\partial M}^p = 1$ and $u \in \mathcal{A}$. Moreover, the weak convergence gives that

$$\|\nabla_g u\|_{p,M}^p \leq \liminf \|\nabla_g u_i\|_{p,M}^p,$$

thus

$$\lambda_{f,K} = \frac{\|\nabla_g u\|_{p,M}^p}{|a|\|u\|_{p,M}^p + |k|\|u\|_{p,\partial M}^p}.$$

which means that $\lambda_{f,K}$ is attained by the function u and by regularity theorems u is $\mathcal{C}^{1,\alpha}(M)$ for certain $\alpha \in (0, 1)$.

Now, we state the following generalized Picone's inequality [1]: for two differentiable functions $u \geq 0$ and $v > 0$, we have

$$(4.1) \quad |\nabla_g u|^p \geq |\nabla_g v|^{p-2} g \left(\nabla_g v, \nabla_g \frac{u^p}{v^{p-1}} \right).$$

Take u a minimizer $\lambda_{f,K}$ and let v be a positive solution of the problem (1.1), then we have

$$\begin{aligned} \int_M |\nabla_g v|^{p-2} g \left(\nabla_g v, \nabla_g \frac{u^p}{v^{p-1}} \right) dv_g &= |a|\|u\|_{p,M}^p + |k|\|u\|_{p,\partial M}^p \\ &+ \int_M f^+ u^p dv_g + \int_{\partial M} K^+ u^p dv_{\sigma_g} \end{aligned}$$

so, by (4.1) we get

$$\lambda_{f,K} = \frac{\|\nabla_g u\|_{p,M}^p}{|a|\|u\|_{p,M}^p + |k|\|u\|_{p,\partial M}^p} \geq 1$$

□

5. SUBCRITICAL PROBLEM: MULTIPLICITY RESULT

In this section, we prove a multiplicity result for problem 3.1, we look for solutions as critical points of the following functional

$$\begin{aligned} E_{q,r}(u) &= \int_M |\nabla_g u|^p dv_g + a \int_M |u|^p dv_g + k \int_{\partial M} |u|^p d\sigma_g \\ &- \frac{p}{q} \int_M f |u|^q dv_g - \frac{p}{r} \int_{\partial M} K |u|^r, q \in [p, p^*], r \in [p, p^{**}]. \end{aligned}$$

which is bounded on the set

$$\mathcal{S}_{\ell,q,r} = \{u \in H_1^p(M) : \|u\|_{q,M}^q + \|u\|_{r,\partial M}^r = \ell\}.$$

set

$$\mu_{\ell,q,r} = \inf_{\mathcal{S}_{\ell,q,r}} E(u)_{q,r}$$

We prove the following lemmas

Lemma 5.1. $\mu_{\ell,q,r}$ is attained for $q < p^*$ and $r < p^{**}$.

Proof. Let $\ell > 0$ and $\{u_i\}_{i \geq 1} \subset \mathcal{S}_{\ell,m,q,r}$ be a minimizing sequence; that is $\|u_i\|_{q,M}^q + \|u_i\|_{r,\partial M}^r = \ell$ and $\lim_{i \rightarrow \infty} E_{q,r}(u_i) = \mu_{\ell,m,q,r}$. For i large enough we can assume $E_{q,r}(u_i) \leq \mu_{\ell,q,r} + 1$, which gives that

$$\begin{aligned} \|\nabla_g u_i\|_{p,M}^p &\leq \mu_{\ell,q,r} + |a| \text{Vol}(M)^{1-\frac{p}{q}} \ell^{\frac{p}{q}} + |k| \text{Vol}(\partial M)^{1-\frac{p}{r}} \ell^{\frac{p}{r}} \\ &\quad + (\sup_M f + \sup_{\partial M} K) \ell + 1. \end{aligned}$$

Thus, the sequence $\{u_i\}_{i \geq 1}$ is bounded in $H_1^p(M)$. Since the inclusions of $H_1^p(M)$ in $L_p(M)$, $L_p(\partial M)$, $L_q(M)$ and $L_r(\partial M)$ are compact for $q < p^*$ and $r < p^{**}$, then there exist a subsequence u_i and a function $u \in H_1^p(M)$ such that u_i converges to u weakly in $H_1^p(M)$ and strongly in each of the spaces $L_p(M)$, $L_p(\partial M)$, $L_q(M)$, $L_r(\partial M)$. It converges also to u almost everywhere in M and in sense of trace on ∂M . Thus, $u \in \mathcal{S}_{\ell,m,q,r}$ and $E_{q,r}(u) \geq \mu_{\ell,m,q,r}$. Moreover, the strong and weak convergence imply that

$$\|\nabla_g u\|_{M,p} \leq \liminf \|\nabla_g u_i\|_{M,p}$$

hence

$$E_{q,r}(u) \leq \lim_{i \rightarrow \infty} E_{q,r}(u_i) = \mu_{\ell,q,r}$$

which means that $\mu_{\ell,m,q,r}$ is attained. \square

Lemma 5.2. $\mu_{\ell,q,r}$ is continuous as a function in the variable ℓ .

Proof. Let $\ell \in]0, \infty[$ and $\{\ell_n\}_{n \in \mathbb{N}} \subset]0, \infty[$ be a sequence such that $\lim_{n \rightarrow \infty} \ell_n = \ell < \infty$. By lemma 5.1, for every $n \in \mathbb{N}$ there exist $u_n \in \mathcal{S}_{\ell_n,q,r}$ such that $E_{q,r}(u_n) = \mu_{\ell_n,q,r}$ and there exists a function $u \in \mathcal{S}_{\ell,q,r}$ such that $E_{q,r}(u) = \mu_{\ell,q,r}$. The sequence u_n is bounded in the spaces $L_q(M)$, $L_r(\partial M)$, $L_p(M)$, $L_p(\partial M)$ and satisfies

$$\begin{aligned} \|\nabla_g u_n\|_{p,M}^p &\leq \mu_{\ell_n,q,r} + |a| \text{vol}(M)^{1-\frac{p}{q}} \ell_n^{\frac{p}{q}} + |k| \text{vol}(\partial M)^{1-\frac{p}{r}} \ell_n^{\frac{p}{r}} \\ &\quad + (\sup_M f + \sup_{\partial M} K) \ell_n. \end{aligned}$$

On the other hand, there exists $t_n > 0$, $\lim_{n \rightarrow \infty} t_n = 1$ such that $t_n u \in \mathcal{S}_{\ell_n,q,r}$. Then

$$\mu_{\ell_n,q,r} \leq t_n^p \|\nabla_g u\|_{p,M}^p + \left(t_n^q \inf_M f + t_n^r \inf_{\partial M} K \right) \ell,$$

which gives that the sequence $\{u_n\}_{n \geq 0}$ is bounded in $H_1^p(M)$. Up to a subsequence u_n converges to a function \hat{u} strongly in the spaces $L_q(M)$, $L_r(\partial M)$, $L_p(M)$, $L_p(\partial M)$ and weakly in $H_1^p(M)$. Thus $\hat{u} \in \mathcal{S}_{\ell,q,r}$ which implies that $E_{q,r}(\hat{u}) \geq E_{q,r}(u)$.

Hence,

$$\liminf_{n \rightarrow \infty} E_{q,r}(u_n) - E_{q,r}(u) \geq 0.$$

On the other hand, we have

$$\begin{aligned} E_{q,r}(u_n) &\leq t_n^p \left(\|\nabla_g u\|_{p,M}^p + a\|u\|_{p,M}^p + k\|u\|_{p,\partial M}^p \right) \\ &\quad - t_n^q \frac{p}{q} \int_M f|u|^q dv_g - t_n \frac{p}{r} \int_{\partial M} K|u|^r d\sigma_g, \end{aligned}$$

thus

$$\lim_{n \rightarrow \infty} \sup E_{q,r}(u_n) - E(u)_{q,r} \leq 0.$$

Therefore, $\mu_{\ell,m,q,r}$ is continuous. \square

Lemma 5.3. $\mu_{\ell,q,r}$ is negative for ℓ small.

Proof. For $\ell > 0$, let $m_\ell > 0$ be the solution of the equation $\text{vol}(M)m_\ell^q + \text{vol}(\partial M)m_\ell^r = \ell$. Take the constant function $u = m_\ell$, then $u \in \mathcal{S}_{\ell,q,r}$ and

$$\begin{aligned} \mu_{\ell,q,r} \leq E_{q,r}(u) &= m_\ell^p \left[-|a|\text{vol}(M) - |k|\text{vol}(\partial M) - \frac{p}{q}m_\ell^{q-p} \int_M f dv_g \right. \\ &\quad \left. - \frac{p}{r}m_\ell^{r-p} \int_{\partial M} K d\sigma_g \right], \end{aligned}$$

If ℓ is small, then so is m_ℓ and we conclude the lemma. \square

Lemma 5.4. $\mu_{\ell,q,r}$ is negative for ℓ big.

Proof. Let u and v be respectively smooth functions defined on M and ∂M with supports included respectively in the sets where $f(x) > 0$ and $K(x) > 0$ and such that $\|u\|_{q,M}^q = 1$ and $\|v\|_{r,\partial M}^r = 1$. For $x \in \overline{M}$, set $h(x) = \frac{1}{2}(\ell^{\frac{1}{q}}u(x) + \ell^{\frac{1}{r}}v(x))$, then $h \in \mathcal{S}_{\ell,q,r}$ and we get

$$\begin{aligned} \mu_{\ell,q,r} \leq E(h) &= \frac{1}{2^p} \ell \left[\left(\|\nabla_g u\|_{p,M}^p + a\|u\|_{p,M}^p \right) \ell^{\frac{p}{q}-1} + k\|u\|_{p,\partial M}^p \ell^{\frac{p}{r}-1} - \right. \\ &\quad \left. \frac{1}{2^{q-p}} \int_M f|u|^q dv_g - \frac{1}{2^{r-p}} \int_{\partial M} K|v|^r d\sigma_g \right]. \end{aligned}$$

since $\int_M f|u|^q dv_g > 0$ and $\int_{\partial M} K|v|^r d\sigma_g > 0$, then for ℓ large enough $\mu_{\ell,q,r} < 0$. \square

Now, let us define the quantity

$$\lambda_{f,K,\eta,q,r} = \inf_{\mathcal{K}(\eta,q,r)} \frac{\|\nabla_g u\|_{p,M}^p}{|a|\|u\|_{p,M}^p + |k|\|u\|_{p,\partial M}^p}$$

where

$$\begin{aligned} \mathcal{K}(\eta, q, r) &= \left\{ u \in H_1^p(M) : \|u\|_{q,M}^q + \|u\|_{r,\partial M}^r = 1 \text{ and} \right. \\ &\quad \left. \int_M |f^-| |u|^q dv_g + \int_{\partial M} |K^-| |u|^r d\sigma_g \leq \eta \left(\int_M |f^-| dv_g + \int_{\partial M} |K^-| d\sigma_g \right) \right\}. \end{aligned}$$

Lemma 5.5. For $(q, r) \in]p, p^*[\times]p, p^{**}[$, $\lambda_{f,K,\eta,q,r}$ converges to $\lambda_{f,K}$ when η goes to zero.

Proof. First let us prove that $\lambda_{f,K,\eta,q,r}$ is attained. Let $u_{i,\eta} \in \mathcal{K}(\eta, q, r)$ be a minimizing sequence for $\lambda_{f,K,\eta,q,r}$, then for i large enough we can have

$$\|\nabla_g u_{i,\eta}\|_{p,M}^p \leq \lambda_{f,K,\eta,q,r} (|a|\|u_{i,\eta}\|_{p,M}^p + |k|\|u_{i,\eta}\|_{p,\partial M}^p) + 1.$$

Hence, the sequence $u_{i,\eta} \in \mathcal{K}(\eta, q, r)$ is bounded in $H_1^p(M)$ and up to a subsequence $u_{i,\eta}$ converges to a function u_η weakly in $H_1^p(M)$ and strongly in each of the spaces $L_p(M), L_p(\partial M), L_q(M), L_r(\partial M)$. Thus, $u_\eta \in \mathcal{K}(\eta, q, r)$. Moreover,

$$\|\nabla_g u_\eta\|_{M,p} \leq \liminf_{i \rightarrow \infty} \|\nabla_g u_{i,\eta}\|_{M,p}.$$

Hence

$$\frac{\|\nabla_g u_\eta\|_{M,p}}{|a|\|u_\eta\|_{p,M}^p + |k|\|u_\eta\|_{p,\partial M}^p} = \lambda_{f,K,\eta,q,r}.$$

Now, consider u_η a sequence of η . First we observe that if $u \in \mathcal{A}$ then there exists $\beta > 0$ such that $\beta u \in \mathcal{K}(\eta, q, r)$. By homogeneity, we get then that the sequence $\lambda_{f,K,\eta,q,r}$ is bounded by $\lambda_{f,K}$. Thus, the sequence u_η is bounded in $H_1^p(M)$ and there exists a subsequence that converges, when η goes to zero, to a function u weakly in $H_1^p(M)$ and strongly in $L_p(M), L_p(\partial M), L_q(M)$ and $L_r(\partial M)$. Hence, $u \in \mathcal{A}$.

On the other hand

$$\|\nabla_g u\|_{p,M}^p \leq \liminf_{\eta \rightarrow 0} \|\nabla_g u_\eta\|_{p,M}^p.$$

Thus

$$\lambda_{f,K} \leq \frac{\|\nabla_g u\|_{p,M}^p}{|a|\|u\|_{p,M}^p + |k|\|u\|_{p,\partial M}^p} \leq \liminf_{\eta \rightarrow 0} \lambda_{f,K,\eta,q,r}.$$

□

Now, coming back to the function $\ell \in (0, \infty) \rightarrow \mu_{\ell,q,r}$ to prove the following lemma

Lemma 5.6. *Suppose that $\lambda_{f,K} > 1$, then there exist two positive constants N_q and H_r such that if $\frac{\sup_M f}{\int_M |f^-| dv_g} \leq N_q$ and $\frac{\sup_{\partial M} K}{\int_{\partial M} |K^-| d\sigma_g} \leq H_r$ then there exists an interval $[\ell_1, \ell_2]$ such that $\mu_{\ell,q,r}$ is positive for every $\ell \in [\ell_1, \ell_2]$.*

Proof. First, if $\lambda_{f,K} > 1$, it follows from lemma 5.5 that for η small enough $\lambda_{f,K,\eta,q,r} - 1 = \frac{\epsilon}{|a|} > 0$.

Let $u \in H_1^p(M)$ be such that $\|u\|_{q,M}^q + \|u\|_{r,\partial M}^r = \ell$ with

$$(5.1) \quad \ell \geq z = \max \left(\left(\frac{2\text{vol}(M)^{1-\frac{p}{q}} |a|}{\eta \int_M |f^-| dv_g} \right)^{\frac{q}{q-p}}, \left(\frac{2\text{vol}(\partial M)^{1-\frac{p}{r}} |k|}{\eta \int_{\partial M} |K^-| d\sigma_g} \right)^{\frac{r}{r-p}} \right),$$

take $\eta > 0$ small enough so that $\ell > 1$.

Denote by $G_{q,r}$ the functional

$$\begin{aligned} G_{q,r}(u) &= \|\nabla_g u\|_{p,M}^p + a\|u\|_{p,M}^p + k\|u\|_{p,\partial M}^p \\ &\quad + \frac{p}{q} \int_M |f^-| |u|^q dv_g + \frac{p}{r} \int_{\partial M} |K^-| |u|^r d\sigma_g. \end{aligned}$$

We distinguish to cases:

either

$$\frac{p}{q} \int_M |f^-| |u|^q dv_g + \frac{p}{r} \int_{\partial M} |K^-| |u|^r d\sigma_g \geq \eta \ell \left(\int_M |f^-| dv_g + \int_{\partial M} |K^-| d\sigma_g \right)$$

and so

$$\begin{aligned}
G_{q,r}(u) &\geq \text{vol}(M)^{1-\frac{p}{q}} a \ell^{\frac{p}{q}} + \text{vol}(\partial M)^{1-\frac{p}{r}} k \ell^{\frac{p}{r}} + \ell \left(\eta \int_M |f^-| dv_g + \eta \int_{\partial M} |K^-| d\sigma_g \right) \\
&\geq \text{vol}(M)^{1-\frac{p}{q}} |a| \ell^{\frac{p}{q}} \left[\frac{\eta \int_M |f^-| dv_g}{\text{vol}(M)^{1-\frac{p}{q}} |a|} \ell^{1-\frac{p}{q}} - 1 \right] + \text{vol}(\partial M)^{1-\frac{p}{r}} |k| \ell^{\frac{p}{r}} \\
&\quad \left[\frac{\eta \int_{\partial M} |K^-| d\sigma_g}{\text{vol}(\partial M)^{1-\frac{p}{r}} |k|} \ell^{1-\frac{p}{r}} - 1 \right] \\
&\geq \text{vol}(M)^{1-\frac{p}{q}} |a| \ell^{\frac{p}{q}} + \text{vol}(\partial M)^{1-\frac{p}{r}} |k| \ell^{\frac{p}{r}}.
\end{aligned}$$

Or

$$\frac{p}{q} \int_M |f^-| |u|^q dv_g + \frac{p}{r} \int_{\partial M} |K^-| |u|^r d\sigma_g \leq \ell \eta \left(\int_M |f^-| dv_g + \int_{\partial M} |K^-| d\sigma_g \right).$$

In this case, let $\delta > 0$ be a solution of the equation $\delta^q \|u\|_{q,M}^q + \delta^r \|u\|_{r,M}^r = 1$, it can be easily seen that $\ell^{-\frac{1}{r}} < \delta < \ell^{-\frac{1}{q}}$. This implies that $\delta u \in \mathcal{K}(\eta, q, r)$. In particular,

$$\|\nabla_g u\|_{p,M}^p \geq \lambda_{f,K,\eta,q,r} \left[|a| \|u\|_{p,M}^p + |k| \|u\|_{p,\partial M}^p \right],$$

so

$$G_{q,r}(u) \geq (\lambda_{f,K,\eta,q,r} - 1) \left[|a| \|u\|_{p,M}^p + |k| \|u\|_{p,\partial M}^p \right],$$

write $\min(|a|, |k|)(\lambda_{f,K,\eta,q,r} - 1) = \alpha_\eta + \beta_\eta$ such that $\frac{\alpha_\eta(|a|+|k|)}{\beta} = \frac{A+B}{K_1+K_2}$, where A, B, K_1, K_2 are the constants appearing in the Sobolev and trace Sobolev inequalities. Then

$$\begin{aligned}
G_{q,r}(u) &\geq \alpha_\eta (\|u\|_{p,M}^p + \|u\|_{p,\partial M}^p) + \frac{\beta}{|a|+|k|} \left[-G_{q,r}(u) + \|\nabla_g u\|_{p,M}^p \right. \\
&\quad \left. + \frac{p}{q} \int_M |f^-| |u|^q dv_g + \frac{p}{r} \int_{\partial M} |K^-| |u|^r d\sigma_g \right]
\end{aligned}$$

that is

$$\left(1 + \frac{\beta_\eta}{|a|+|k|} \right) G_{q,r}(u) \geq \frac{\beta_\eta}{|a|+|k|} \left[\|\nabla_g u\|_{p,M}^p + \frac{\alpha_\eta(|a|+|k|)}{\beta_\eta} (\|u\|_{p,M}^p + \|u\|_{p,\partial M}^p) \right].$$

Then, by the Sobolev and trace sobolev inequalities we get

$$\begin{aligned}
\left(1 + \frac{\beta_\eta}{|a|+|k|} \right) G_{q,r}(u) &\geq \frac{\beta_\eta}{|a|+|k|} \left[\|\nabla_g u\|_{p,M}^p + \frac{A+B}{K_1+K_2} (\|u\|_{p,M}^p + \|u\|_{p,\partial M}^p) \right] \\
&\geq \frac{\beta_\eta}{(|a|+|k|)(K_1+K_2)} (\ell^{\frac{p}{q}} + \ell^{\frac{p}{r}}).
\end{aligned}$$

Thus

$$G(u) \geq \frac{\beta_\eta}{(K_1+K_2)(|a|+|k|+\beta_\eta)} (\ell^{\frac{p}{q}} + \ell^{\frac{p}{r}}).$$

On the other hand, we have

$$\begin{aligned}
E_{q,r}(u) &= G_{q,r}(u) - \int_M f^+ |u|^q dv_g - \int_{\partial M} K^+ |u|^r d\sigma \\
&\geq G_{q,r}(u) - \left(\sup_M f + \sup_{\partial M} K \right) \ell \\
&\geq t \ell^{\frac{p}{q}} + s \ell^{\frac{p}{r}} - \left(\sup_M f + \sup_{\partial M} K \right) \ell,
\end{aligned}$$

where

$$t = \min \left(\frac{\beta_\eta}{(K_1 + K_2)(|a| + |k| + \beta_\eta)}, |a| \text{vol}(M)^{1-\frac{p}{q}} \right),$$

and

$$s = \min \left(\frac{\beta_\eta}{(K_1 + K_2)(|a| + |k| + \beta_\eta)}, |k| \text{vol}(\partial M)^{1-\frac{p}{r}} \right).$$

Now, the function $g : \ell \rightarrow \frac{t}{2}\ell^{\frac{p}{q}} + \frac{s}{2}\ell^{\frac{p}{r}} - (\sup_M f + \sup_{\partial M} K)\ell$ attains a positive maximum in the interval $]0, \ell_o[$ where $g(\ell_o) = 0$. Therefore, the hypotheses on the functions f and K assure that $g(z) > 0$, where z is defined by (5.1). In fact,

$$g(z) \geq z \left[\frac{t\eta \int_M |f^-| dv_g}{4\text{vol}(M)^{1-\frac{p}{q}}|a|} - \sup_M f + \frac{s\eta \int_{\partial M} |K^-| dv_g}{4\text{vol}(\partial M)^{1-\frac{p}{r}}|k|} - \sup_{\partial M} K \right]$$

so if we require that

$$\frac{\sup_M f}{\int_M |f^-| dv_g} \leq N_q = \frac{t\eta}{8\text{vol}(M)^{1-\frac{p}{q}}|a|}$$

and

$$\frac{\sup_{\partial M} K}{\int_{\partial M} |K^-| dv_g} \leq H_r = \frac{s\eta}{8\text{vol}(\partial M)^{1-\frac{p}{r}}|k|}$$

we get $g(z) > 0$ and then $z < \ell_o$. Thus if $\ell \in [z, \ell_o]$, then

$$E_{q,r}(u) > \frac{1}{2}(t\ell^{\frac{p}{q}} + s\ell^{\frac{p}{r}}) > 0,$$

and so $\mu_{\ell,q,r} > 0$ for all $\ell \in [\ell_1, \ell_2] = [z, \ell_o]$. \square

Now, by mean of the mountain pass theorem, we show that the existence of the interval $[\ell_1, \ell_2]$ leads to the existence of a second critical point of the functional $E_{q,r}$. First, we prove the following lemma

Lemma 5.7. *The Plais-Smale condition is satisfied for the functional $E_{q,r}$, $q < p^*$ and $r < p^{**}$.*

Proof. First, we claim that each Plais-Smale sequence for the functional $E_{q,r}$ is bounded in $H_1^p(M)$. In fact, let $u_n \in H_1^p(M)$ be a sequence such that $E_{q,r}(u_n) \xrightarrow{n \rightarrow \infty} \gamma$ and $E'_{q,r}(u_n) \xrightarrow{n \rightarrow \infty} 0$, then we have

$$\begin{aligned} E_{q,r}(u_n) - \frac{1}{q} E'_{q,r}(u_n) u_n &= \left(1 - \frac{p}{q}\right) \left[\|\nabla_g u_n\|_{p,M}^p + a \|u_n\|_{p,M}^p + k \|u_n\|_{p,M}^p \right] \\ &\quad - \left(\frac{1}{r} - \frac{p}{q}\right) \int_{\partial M} K |u_n|^r d\sigma_g \end{aligned}$$

and

$$E_{q,r}(u_n) - \frac{1}{p} E'_{q,r}(u_n) u_n = \left(1 - \frac{p}{q}\right) \int_M f |u_n|^q dv_g + \left(1 - \frac{p}{r}\right) \int_{\partial M} K |u_n|^r d\sigma_g$$

so, for every $\varepsilon > 0$, there exists n_o such that for all $n \geq n_o$, we have

$$\left| \left(1 - \frac{p}{q}\right) \int_M f |u_n|^q dv_g + \left(1 - \frac{p}{q}\right) \int_{\partial M} K |u_n|^r d\sigma_g - \gamma \right| \leq \varepsilon + o(\|u_n\|_{H_1^p(M)})$$

and

$$\begin{aligned} \left| \left(1 - \frac{p}{q}\right) \left[\|\nabla_g u_n\|_{p,M}^p + a \|u_n\|_{p,M}^p + k \|u_n\|_{p,M}^p \right] - \left(\frac{1}{r} - \frac{p}{r}\right) \int_{\partial M} K |u_n|^r d\sigma_g - \gamma \right| \\ \leq \varepsilon + o(\|u_n\|_{H_1^p(M)}) \end{aligned}$$

Let $\ell > 0$ be such that $\mu_{\ell,q,r} > 0$ and put $v_n = \beta_n u_n$, where $\beta_n > 0$ is such that $\beta_n^q \|u\|_{q,M}^q + \beta_n^r \|u\|_{r,M}^r = \ell$. We observe that v_n is bounded in $L_q(M)$ and $L_r(\partial M)$. Now, we have

$$(5.2) \quad \left| \frac{1}{\beta_n^q} \left(1 - \frac{p}{q}\right) \int_M f |v_n|^q dv_g + \frac{1}{\beta_n^r} \left(1 - \frac{p}{r}\right) \int_{\partial M} K |v_n|^r d\sigma_g - \gamma \right| \leq \varepsilon + o(\|v_n\|_{H_1^p(M)})$$

and

$$(5.3) \quad \left| \frac{1}{\beta_n^p} \left(1 - \frac{p}{q}\right) \left[\|\nabla_g v_n\|_{p,M}^p + a \|v_n\|_{p,M}^p + k \|v_n\|_{p,M}^p \right] - \frac{1}{\beta_n^r} \left(\frac{1}{r} - \frac{p}{q}\right) \int_{\partial M} K |v_n|^r d\sigma_g - \gamma \right| \leq \varepsilon + o(\|v_n\|_{H_1^p(M)}).$$

Since the sequence v_n is bounded in $L_q(M)$ and $L_r(\partial M)$, then by (5.3) it is bounded in $H_1^p(M)$. Moreover, we affirm that the sequence u_n is bounded in $L_q(M)$ and $L_r(\partial M)$. In fact, if the sequence u_n goes to infinity in $L_q(M)$ or $L_r(\partial M)$ then the sequence β_n goes to zero when n goes to infinity. This implies by mean of inequalities (5.2) and (5.3) that $E_{q,r}(v_n) \rightarrow 0$ as $n \rightarrow \infty$.

Since $v_n \in \mathcal{S}_{\ell,m,q,r}$, then $E_{q,r}(v_n) \geq \mu_{\ell,q,r} > 0$, this is a patent contradiction. Thus, u_n is bounded in $L_q(M)$ and $L_r(\partial M)$ and since $E'_{q,r}(u_n)u_n \rightarrow 0$ then, it is bounded in $H_1^p(M)$.

Thus, up to a subsequence u_n converges to a function u weakly in $H_1^p(M)$ and strongly in $L_q(M), L_p(M), L_r(\partial M)$ and $L_p(\partial M)$. Then, by Brezis-Lieb lemma, we obtain

$$\begin{aligned} \|\nabla_g(u_n - u)\|_{p,M} &\leq |a| \|u_n - u\|_{p,M}^p + |k| \|u_n - u\|_{p,\partial M}^p + \sup_M f \|u_n - u\|_{q,M}^q \\ &\quad + \sup_{\partial M} K \|u_n - u\|_{r,\partial M}^r + o(1) \\ &\leq o(1). \end{aligned}$$

which means that the subsequence u_n converges strongly to u in $H_1^p(M)$. \square

Now, we prove theorem 3.2

Proof of theorem 3.2.

• Existence of first solution

Let $\ell_1 > 0$ be such that $\mu_{\ell_1,q,r} = 0$ and the curve $\ell \rightarrow \mu_{\ell,q,r}$ is negative for $\ell \in]0, \ell_1[$. Set

$$\mu_{\ell,q,r,q,r} = \inf_{\mathcal{D}_{\ell,q,r}} E_{q,r}$$

where

$$\mathcal{D}_{\ell,q,r} = \{u \in H_1^p(M) : u > 0, \|u\|_{q,M}^q + \|u\|_{r,\partial M}^r \leq \ell, \ell < \ell_1\}$$

Take ℓ as small as $\mu_{\ell,q,r} < 0$, then there exists $u \in \mathcal{D}_{\ell,q,r}$ such that $\|u\|_{q,M}^q + \|u\|_{r,\partial M}^r = \ell$ and $E_{q,r}(u) = \mu_{\ell,q,r}$ in such way that, $u \in \mathcal{D}_{\ell,q,r}$ and

$$(5.4) \quad \mu_{\ell,q,r,q,r} \leq E_{q,r}(u) = \mu_{\ell,q,r} < 0.$$

By using the Ekeland Variational Principle, in the set $\mathcal{D}_{\ell,q,r}$ we can find a sequence $\{u_{q,r,n}\}_n$ such that $E_{q,r}(u_{q,r,n}) \xrightarrow{n \rightarrow \infty} \mu_{\ell,q,r,q,r}$ and $E'_{q,r}(u_{q,r,n}) \xrightarrow{n \rightarrow \infty} 0$. Obviously, the sequence $u_{q,r,n}$ is bounded in $H_1^p(M)$, then up to a subsequence, $u_{q,r,n}$ converges to a function $u_{q,r}$ weakly in $H_1^p(M)$ and strongly in $L_s(M), L_t(\partial M)$ ($s < p^*, t < p^{**}$)

, almost everywhere on M and in the sense of trace on ∂M . Thus, $u_{q,r} \in \mathcal{D}_{\ell,q,r}$ and

$$E_{q,r}(u_{q,r}) \leq \liminf_{n \rightarrow \infty} E_{q,r}(u_{q,r,n}) = \mu_{\ell,q,r,q,r} < 0.$$

Furthermore, by the weak convergence in $H_1^p(M)$, $L_{\frac{q}{q-1}}(M)$ and $L_{\frac{r}{r-1}}(\partial M)$, it follows that for every $v \in H_1^p(M)$

$$\langle E'(u_{q,r}), v \rangle = \lim_{n \rightarrow \infty} \langle E'(u_{q,r,n}), v \rangle = 0.$$

Hence, $u_{q,r}$ is critical point of $E_{q,r}$ with $E_{q,r}(u_{q,r}) < 0$.

• Existence of second solution

Now, we prove that there exists a second solution $v_{q,r}$ with $E_{q,r}(v_{q,r}) > 0$. Let ℓ_1 and ℓ_2 be such that

$$\begin{aligned} \mu_{\ell_1,q,r} = E_{q,r}(u_{\ell_1}) &= 0 \\ \mu_{\ell_2,q,r} = E_{q,r}(u_{\ell_2}) &= 0 \end{aligned}$$

and consider

$$\mu_{q,r} = \inf_{g \in \Gamma} \max_{s \in [0,1]} E_{q,r}(g(s))$$

where

$$\Gamma = \{g \in \mathcal{C}([0,1], H_1^p(M)) : g(0) = u_{\ell_1}, g(1) = u_{\ell_2}\}$$

We claim that $\mu_{q,r}$ is critical value of the functional $E_{q,r}$. In fact, if it is not, then there exists $\varepsilon > 0$ small such that $E_{q,r}$ does not possess any critical value in the interval $[\mu_{q,r} - \varepsilon, \mu_{q,r} + \varepsilon]$. Thus, by the deformations Lemma we can find a function $\phi_t : H_1^p(M) \longrightarrow H_1^p(M)$, $t \in [0,1]$, continuous in t such that:

- (1) $\phi_0(u) = u, \forall u \in H_1^p(M)$
- (2) $\phi_t(u) = u, \forall u \in H_1^p(M)$ such that $E_{q,r}(u) \notin [\mu_{q,r} - \varepsilon, \mu_{q,r} + \varepsilon]$
- (3) $\forall u \in H_1^p(M)$, s.t $E_{q,r}(u) \leq \mu_{q,r} + \varepsilon$, then $E_{q,r}(\phi_1(u)) \leq \mu_{q,r} - \varepsilon$.

Now, let $g \in \Gamma$ be such that $\max_{s \in [0,1]} E_{q,r}(g(s)) \leq \mu_{q,r} + \varepsilon$. By definition of u_{ℓ_1}, u_{ℓ_2} and property (2) of the function ϕ_t , we have

$$\begin{aligned} \phi_t(u_{\ell_1}) &= u_{\ell_1} \\ \phi_t(u_{\ell_2}) &= u_{\ell_2} \end{aligned}$$

In particular, the curve $\phi_1(g) \in \Gamma$ which gives that $\mu_{q,r} \leq \max_{s \in [0,1]} E_{q,r}(\phi_1(g(s)))$, but by property (3), we have that $\max_{s \in [0,1]} E_{q,r}(\phi_1(g(s))) \leq \mu_{q,r} - \varepsilon$, which makes a contradiction. $\mu_{q,r}$ is then a critical level for the functional $E_{q,r}$ and

$$\mu_{q,r} > \sup_{\ell \in [\ell_1, \ell_2]} \mu_{\ell,q,r} > 0$$

Therefore, there exists $v_{q,r}$ a solution of (3.1) with $E_{q,r}(v_{q,r}) = \mu_{q,r} > 0$ and theorem 3.2 is proven. \square

6. CRITICAL PROBLEM: MULTIPLICITY RESULT

In this section, we prove existence and multiplicity of solutions of the problem (1.1). We will study the limits of the sequences $u_{q,r}$ and $v_{q,r}$ as (q, r) goes to (p^*, p^{**}) . Here, besides the non-compactness of the inclusions $H_1^p(M) \hookrightarrow L_{p^*}(M)$ and $L_{p^{**}}(\partial M)$, due to the fact that the functions f and K change the sign, we face serious problems in proving distinction among the limits. The curve $\ell \rightarrow \mu_{\ell,q,r}$ will play an important role in overcoming these problems.

Let E be the functional

$$E = \int_M |\nabla_g u|^p dv_g + a \int_M |u|^p dv_g + k \int_{\partial M} |u|^p d\sigma_g \\ - \frac{n-p}{n} \int_M f |u|^{p^*} dv_g - \frac{n-p}{n-1} \int_{\partial M} K |u|^{p^{**}} d\sigma_g,$$

6.1. Existence of first solution.

Proof of theorem 3.1. Let $u_{q,r} > 0$ be the sequence of critical points of the functional $E_{q,r}$ such that $E_{q,r}(u_{q,r}) < 0$. This sequence is bounded in $H_1^p(M)$. In fact, we have

$$\|u_{q,r}\|_{q,M}^q + \|u_{q,r}\|_{r,\partial M}^r = \ell_{q,r} < \ell_1 \\ < \max \left(\left(\frac{2\text{vol}(M)^{1-\frac{p}{n}}|a|}{\eta \int_M |f^-| dv_g} \right)^{\frac{n}{p}}, \left(\frac{2\text{vol}(\partial M)^{1-\frac{p}{n-1}}|k|}{\eta \int_{\partial M} |K^-| d\sigma_g} \right)^{\frac{n-1}{p}} \right) + \varepsilon$$

thus $u_{q,r}$ is bounded in $L_q(M)$ and $L_r(\partial M)$. Moreover, by (5.4) we have

$$\|\nabla_g u_{q,r}\|_{p,M} \leq \mu_{r,q} + |a| \text{vol}(M)^{\frac{p}{n}} \ell_1^{\frac{n-p}{n}} + |k| \text{vol}(\partial M)^{\frac{p-1}{n-1}} \ell_1^{\frac{n-p}{n-1}} \\ + (\sup_M f + \sup_{\partial M} K) \ell_1.$$

Since $\mu_{\ell,q,r} < 0$, the sequence $u_{q,r}$ is bounded in $H_1^p(M)$. Then, we can obtain a subsequence $u_{q,r}$ and a function $u \in H_1^p(M)$ such that

- (1) $u_{q,r}$ converges weakly to u in $H_1^p(M)$
- (2) $u_{q,r}$ converges strongly to u in $L_p(M)$, $L_p(\partial M)$, $L_{p^*-1}(M)$ and $L_{p^{**}-1}(\partial M)$
- (3) $u_{q,r}$ converges almost everywhere to u in M and in the sens of trace on ∂M .
- (4) The sequence $\nabla_g u_{q,r}$ converges almost everywhere to $\nabla_g u$

Thus, u is critical point of E . By Brezis-Lieb lemma [13], Sobolev and trace Sobolev inequalities, we get

$$E_{q,r}(u_{q,r}) - E_{q,r}(u) = \|\nabla_g(u_{q,r} - u)\|_{p,M}^p - (1 - \frac{p}{q}) \int_M f |u_{q,r} - u|^q dv_g \\ - (1 - \frac{p}{q}) \int_{\partial M} K |u_{q,r} - u|^r d\sigma_g + o(1) \\ \geq \frac{1}{K_1 + K_2} \left[\|u_{q,r} - u\|_{q,M}^p + \|u_{q,r} - u\|_{r,\partial M}^p \right] \\ - \sup_M f \|u_{q,r} - u\|_{q,M}^q - \sup_{\partial M} K \|u_{q,r} - u\|_{r,\partial M}^r + o(1)$$

then, taking into account, by Brezis-Lieb Lemma again that $\lim_{(q,r) \rightarrow (p^*, p^{**})} \|u_{q,r} - u\|_{q,M}^q + \|u_{q,r} - u\|_{r,\partial M}^r \leq \ell, \ell > 0$, we get

$$\begin{aligned} E(u) &\leq \lim_{(q,r) \rightarrow (p^*, p^{**})} E_{q,r}(u_{q,r}) - \frac{1}{K_1 + K_2} \lim_{(q,r) \rightarrow (p^*, p^{**})} \left[\|u_{q,r} - u\|_{q,M}^p \right. \\ &\quad \left. + \|u_{q,r} - u\|_{r,M}^p \right] + (\sup_M f + \sup_{\partial M} K) \ell. \end{aligned}$$

Sine, a priori, we have that $\lim_{(q,r) \rightarrow (p^*, p^{**})} \left[\|u_{q,r} - u\|_{q,M}^p + \|u_{q,r} - u\|_{r,M}^p \right] > 0$, by taking ℓ small enough, we obtain $E(u) < 0$, thus $u \neq 0$ and we are done. \square

6.2. Existence of second solution. Now, we consider the sequence $\{v_{q,r}\}$ of solutions of the subcritical problem (3.1) obtained by the mountain pass lemma. We prove that $\{v_{q,r}\}$ will converge to non zero and different critical point of E . We recall that the sequence $\{v_{q,r}\}$ fulfills the following properties

$$(6.1) \quad E'_{q,r}(v_{q,r}) = 0 \text{ and } E_{q,r}(v_{q,r}) = \mu_{q,r} > 0.$$

First, we prove the following lemma

Lemma 6.1. *The sequence of functions $v_{q,r}$ is bounded in $H_1^p(M)$.*

Proof. Let $\ell_1 > 0$ and $\ell_2 > 0, \ell_1 < \ell_2$ be two real numbers with the associated functions u_{ℓ_1} and u_{ℓ_2} such that

$$\begin{aligned} \mu_{\ell_1,q,r} = E_{q,r}(u_{\ell_1}) &= 0 \\ \mu_{\ell_2,q,r} = E_{q,r}(u_{\ell_2}) &= 0. \end{aligned}$$

For $s \in [0, 1]$, let g be the curve $g(s) = su_{\ell_1} + (1-s)u_{\ell_2}$. Then by definition of $\mu_{q,r}$ we have

$$\mu_{q,r} \leq \max_{s \in [0,1]} E_{q,r}(g(s)).$$

Since $E_{q,r}(g(1)) = 0$ and we can find $s \in [0, 1]$ such that

$$\|g(s)\|_{q,M}^q + \|g(s)\|_{r,\partial M}^r = \frac{\ell_1 + \ell_2}{2} > \ell_1,$$

it follows that the curve $E_{q,r}(g(s))$ attains for certain $s_o \in (0, 1)$ a positive maximum. Thus

$$\begin{aligned} \mu_{q,r} &\leq E_{q,r}(g(s_o)) = (1 - \frac{p}{q}) \int_M f |g(s_o)|^q dv_g + (1 - \frac{p}{r}) \int_{\partial M} K |g(s_o)|^r d\sigma_g \\ &\leq (\sup_M f + \sup_{\partial M} K) (\ell_1 + \ell_2). \end{aligned}$$

which gives that the sequence $\mu_{q,r}$ is uniformly bounded in (q, r) .

Now, it remains to show that the sequence $v_{q,r}$ is bounded in $L_q(M)$ and $L_r(\partial M)$. We proceed as in the proof of theorem 3.2. Let $\ell_o \in (\ell_1, \ell_2)$ be such that $\mu_{\ell_o,q,r} = \sup_{\ell} \mu_{\ell,q,r} > 0$ and consider the sequence $u_{q,r} = \beta_{q,r} v_{q,r}$ where $\beta_{q,r}$ is such that $\beta_{q,r}^q \|v_{q,r}\|_{q,M}^q + \beta_{q,r}^r \|v_{q,r}\|_{r,\partial M}^r = \ell_o$. The sequence $u_{q,r}$, such as defined, satisfies

$$(6.2) \quad \mu_{q,r} = (1 - \frac{p}{q}) \beta_{q,r}^{-q} \int_M f |u_{q,r}|^q dv_g + (1 - \frac{p}{r}) \beta_{q,r}^{-r} \int_{\partial M} K |u_{q,r}|^r d\sigma_g,$$

and

$$\begin{aligned} (6.3) \quad E_{q,r}(u_{q,r}) &= \beta_{q,r}^p \mu_{q,r} + \frac{p}{q} (\beta_{q,r}^{p-q} - 1) \int_M f |u_{q,r}|^q dv_g \\ &\quad + \frac{p}{r} (\beta_{q,r}^{p-r} - 1) \int_{\partial M} K |u_{q,r}|^r d\sigma_g. \end{aligned}$$

Suppose by contradiction that the sequence $v_{q,r}$ goes to infinity in $L_q(M)$ and $L_r(\partial M)$ as (q, r) goes to (p^*, p^{**}) . Then, the sequence $\beta_{q,r}$ should go to zero. Since $\mu_{q,r}$ is bounded, we get necessarily that $\int_M f|u_{q,r}|^q dv_g$ and $\int_{\partial M} K|u_{q,r}|^r d\sigma_g$ go to zero as (q, r) goes to (p^*, p^{**}) and $\beta_{q,r}^{-q} \int_M f|u_{q,r}|^q dv_g$, $\beta_{q,r}^{-r} \int_{\partial M} K|u_{q,r}|^r d\sigma_g$ are both bounded. Thus, by (6.3), we get that $E_{q,r}(u_{q,r})$ goes to zero as (q, r) goes to (p^*, p^{**}) .

On the other hand, we have $E_{q,r} > \mu_{\ell_o, q, r}$ and by lemma 5.6 the sequence $\mu_{\ell_o, q, r}$ does not go to zero as (q, r) goes to (p^*, p^{**}) , this makes a contradiction. Therefore, the sequence $v_{q,r}$ is bounded in $L_q(M)$ and $L_r(\partial M)$ and since it satisfies

$$\begin{aligned} \|\nabla_g v_{q,r}\|_{p,M}^p &= |a| \|v_{q,r}\|_{p,M}^p + |k| \|v_{q,r}\|_{p,\partial M}^p \\ &+ \int_M f|v_{q,r}|^q dv_g + \int_{\partial M} K|u_{q,r}|^r d\sigma_g, \end{aligned}$$

then it is bounded in $H_1^p(M)$. \square

Now, as the sequence $v_{q,r}$ is bounded in $H_1^p(M)$, we can extract a subsequence that converges strongly to a function v in $L_p(M)$ and $L_p(\partial M)$ and weakly in $H_1^p(M)$, $L_{\frac{p^*}{p^*-1}}(M)$ and $L_{\frac{p^{**}}{p^{**}-1}}(\partial M)$. The function v is then a critical point of the functional E . But, this is not enough to conclude existence of second solution because in spite of the fact that $\lim_{(q,r) \rightarrow (p^*, p^{**})} E_{q,r}(v_{q,r}) - E_{q,r}(u_{q,r}) > 0$, we could have $v = u$ or $v = 0$ regarding the lack of the strong convergence of the sequence $v_{q,r}$ to v in $L_{p^*}(M)$ and $L_{p^{**}}(\partial M)$. In the following lemmas, we give sufficient conditions to prevent such cases from occurrence.

First, note that the first solution u satisfies that $\|u\|_{p^*, M}^{p^*} + \|u\|_{p^{**}, \partial M}^{p^{**}} \leq \ell < \ell_1$. Take ℓ such that

$$\ell < \frac{(\sup_M f)^{-\frac{n-p}{p}} K_1^{-n}}{|\inf_M f| + |\inf_{\partial M} K|},$$

we get then

$$E(u) + \frac{p}{n} \left(\sup_M f \right)^{1-\frac{n}{p}} K_1^{-n} > 0.$$

Let us prove the following lemma

Lemma 6.2. *Suppose that the sequence $v_{q,r}$ converges strongly to the function u in $L_p(M)$ and $L_p(\partial M)$ and that the functions f and K satisfy*

$$\frac{p(n-1)}{n(p-1)} (\sup_M f)^{-\frac{n-p}{p}} (\sup_{\partial M} K)^{\frac{n-p}{p-1}} K_1^{-n} K_2^{\frac{p(n-1)}{p-1}} \leq 1$$

If the following condition is satisfied

$$(6.4) \quad \lim_{(q,r) \rightarrow (p^*, p^{**})} \mu_{q,r} < E(u) + \frac{p}{n} \left(\sup_M f \right)^{1-\frac{n}{p}} K_1^{-n}$$

then the sequence $v_{q,r}$ converges strongly to u in $H_1^p(M)$.

Proof. The sequence $v_{q,r}$ is bounded. We may assume that $v_{q,r}$ converges to u weakly in $H_1^p(M)$, almost every where on M and in the sense of trace on ∂M .

In particular, we assume that the sequences of measures $|\nabla_g v_{q,r}|^p dv_g$, $|v_{q,r}|^{p^*} dv_g$ and $|v_{q,r}|^{p^{**}} d\sigma_g$ converge weakly in the sense of measures respectively to bounded nonnegative measures $d\mu$, $d\nu$ and $d\pi$.

Thus, by a version of a concentration-compactness theorem for manifolds with

boundary [11], there exist at most countable index set I , sequence of points $\{x_i\}_{i \in I} \subset \overline{M}$ and positive numbers $\{\mu_i\}_{i \in I}, \{\nu_i\}_{i \in I}, \{\pi_i\}_{i \in I}$ such that

$$\begin{aligned} d\mu &\geq |\nabla_g u|^p dv_g + \sum_{i \in I} \mu_i \delta_{x_i}, \\ d\nu &= |u|^{p^*} dv_g + \sum_{i \in I} \nu_i \delta_{x_i}, \text{ and} \\ d\pi &= |u|^{p^{**}} d\sigma_g + \sum_{i \in J} \pi_i \delta_{y_i}. \end{aligned}$$

Moreover,

$$\nu_i^{\frac{1}{p^*}} \leq K_1 \mu_i^{\frac{1}{p}} \text{ and } \pi_i^{\frac{1}{p^{**}}} \leq K_2 \mu_i^{\frac{1}{p}}.$$

Take $x_i \in \overline{M}$ in the support of the singular part of μ, ν and π and for $\varepsilon > 0$, let ϕ be a $\mathcal{C}^\infty(B(x_i, \varepsilon))$ cut-off function such that $\text{supp } \phi \subset (B(x_i, \varepsilon), \phi \equiv 1 \text{ on } (B(x_i, \frac{\varepsilon}{2})))$ and $|\nabla_g \phi| \leq C$. Then, we get

$$\begin{aligned} \int_{B(x_i, \varepsilon)} \phi d\mu &= \lim_{(q, r) \rightarrow (p^*, p^{**})} \int_{B(x_i, \varepsilon)} |\nabla_g v_{q, r}^{p-2}| g(\nabla_g v_{q, r}, \nabla_g \phi) dv_g \\ &= -a \int_{B(x_i, \varepsilon)} |u|^{p-2} u \phi dv_g - k \int_{B(x_i, \varepsilon)} |u|^{p-2} u \phi d\sigma_g \\ &\quad + \int_{B(x_i, \varepsilon)} f \phi d\nu + \int_{B(x_i, \varepsilon)} K \phi d\pi \end{aligned}$$

this implies, after letting ε tends to zero, that for each $i \in J$, depending on whether $x_i \in M$ or $x_i \in \partial M$,

$$\mu_i \leq f(x_i) \nu_i \text{ or } \mu_i \leq K(x_i) \pi_i.$$

In particular, we assume that either $f(x_i) > 0$ or $K(x_i) > 0$, because otherwise we get $\mu_i = \nu_i = \pi_i = 0$ and we are done.

Now, suppose that there exists $i \in I$ such that $\nu_i \neq 0$ or $\pi_i \neq 0$ (depending on whether x_i is in M or ∂M), then we get

$$\begin{aligned} \lim_{(q, r) \rightarrow (p^*, p^{**})} \mu_{q, r} &= \lim_{(q, r) \rightarrow (p^*, p^{**})} \left(\left(1 - \frac{p}{q}\right) \int_M f v_{q, r}^q dv_g + \left(1 - \frac{p}{r}\right) \int_{\partial M} K v_{q, r}^r d\sigma_g \right) \\ &= \frac{p}{n} \int_M f u^{p^*} dv_g + \frac{p-1}{n-1} \int_{\partial M} K u^{p^{**}} d\sigma_g + \frac{p}{n} f(x_i) \nu_i + \frac{p-1}{n-1} K(x_i) \pi_i \\ &\geq E(u) + \frac{p}{n} \left[\sup_M f \right]^{1-\frac{n}{p}} K_1^{-n}, \end{aligned}$$

which contradicts the hypothesis of the lemma. Thus, we get $\mu_i = \nu_i = \pi_i = 0$ and the sequence $v_{q, r}$ converges strongly to u in $L_{p^*}(M)$ and $L_{p^{**}}(\partial M)$. \square

In the following, we give a sufficient condition in order to get satisfied condition (6.4) of lemma 6.2.

Lemma 6.3. *Suppose that there exists a positive function $\Phi \in H_1^p(M)$ such that $I(\Phi) > 0$, $\int_M f \Phi^{p^*} dv_g > 0$, $\int_{\partial M} K \Phi^{p^{**}} d\sigma_g > 0$ and*

$$0 < \sup_{\lambda \in [0, 1]} E(\lambda \Phi) < E(u) + \frac{p}{n} \left[\sup_M f \right]^{1-\frac{n}{p}} K_1^{-n}$$

then condition (6.4) of lemma 6.2 is satisfied.

Proof. Let u_{ℓ_1}, u_{ℓ_2} be such that $E_{q,r}(u_{\ell_1}) = E_{q,r}(u_{\ell_2}) = 0$ and $\ell_1 = \|u_{\ell_1}\|_{q,M}^q + \|u_{\ell_1}\|_{r,\partial M}^r < \ell_o < \ell_2 = \|u_{\ell_2}\|_{q,M}^q + \|u_{\ell_2}\|_{r,\partial M}^r$, where ℓ_o is such that $\mu_{\ell_o} = \sup_{\ell \in]\ell_1, \ell_2[} \mu_{q,r,\ell}$. Then, the curve $\lambda \rightarrow E_{q,r}(\frac{\alpha\lambda-1}{\alpha-1}u_{\ell_2})$, $\alpha > 1, \lambda \in [\frac{1}{\alpha}, \infty[$, starts from zero, increases towards a positive maximum for $\lambda < 1$ and then decreases to minus infinity.

Now, suppose that there exists a function $\Phi \in H_1^p(M)$ such that $\|\Phi\|_{q,M}^q + \|\Phi\|_{r,\partial M}^r > \ell_2$. Let $\delta_o > \delta_1 > 1$ be two constants and consider the the curve $\lambda \rightarrow E_{q,r}((1 - \delta_o\lambda)(\delta_1\lambda - 1)\Phi)$, $\lambda \in [\frac{1}{\delta_1}, \infty[$, then we have,

$$\begin{aligned} \frac{dE_{q,r}((1 - \delta_o\lambda)(\delta_1\lambda - 1)\Phi)}{d\lambda} &= p(\delta_1 + \delta_o) [|1 - \delta_o\lambda| |\delta_1\lambda - 1|]^{p-1} (1 - \frac{2\delta_o\delta_1}{\delta_1 + \delta_o}\lambda) \\ &\quad \left[\|\nabla_g \Phi\|_{p,M}^p + a\|\Phi\|_{p,M}^p + k\|\Phi\|_{p,\partial M}^p - (|1 - \delta_o\lambda| |\delta_1\lambda - 1|)^{q-p} \int_M f\Phi^q dv_g \right. \\ &\quad \left. - (|1 - \delta_o\lambda| |\delta_1\lambda - 1|)^{r-p} \int_{\partial M} K\Phi^r dv_g \right]. \end{aligned}$$

Note $F_\Phi(\lambda)$ the function

$$\begin{aligned} F_\Phi(\lambda) &= I(\phi) - ((1 - \delta_o\lambda)(\delta_1\lambda - 1))^{q-p} \int_M f\Phi^q dv_g \\ &\quad - ((1 - \delta_o\lambda)(\delta_1\lambda - 1))^{r-p} \int_{\partial M} K\Phi^r d\sigma_g \end{aligned}$$

with $\lambda \in [\frac{1}{\delta_o}, \frac{1}{\delta_1}]$, $I(\phi) > 0$, $\int_M f\Phi^q dv_g > 0$ and $\int_{\partial M} K\Phi^r d\sigma_g > 0$.

Then, in the interval $(\frac{1}{\delta_o}, \frac{1}{\delta_1})$, there exist at most two values $\frac{1}{\delta_2} > \frac{\delta_o + \delta_1}{2\delta_o\delta_1} > \frac{1}{\delta_3}$ of λ such that $F_\Phi(\frac{1}{\delta_2}) = F_\Phi(\frac{1}{\delta_3}) = 0$, $F_\Phi(\lambda) > 0$, $\lambda \in (\frac{1}{\delta_o}, \frac{1}{\delta_3}) \cup (\frac{1}{\delta_2}, \frac{1}{\delta_1})$ and $F_\Phi(\lambda) < 0$, $\lambda \in (\frac{1}{\delta_3}, \frac{1}{\delta_2})$.

Hence, there exists $\delta_o > \delta'_o > \delta_1$ such that the curve $\lambda \rightarrow E_{q,r}((1 - \delta_o\lambda)(\delta_1\lambda - 1)\Phi)$ is positive for $\lambda \in [\frac{1}{\delta'_o}, \frac{1}{\delta_1}]$ and attains positive maximum at $\lambda_o = \frac{1}{\delta_2}$.

On the other hand, the curve $\lambda \rightarrow E_{q,r}((1 - \delta_o\lambda)u_{\ell_1})$, starts from zero, decreases to negative minimum in the interval $(0, \frac{1}{\delta_o})$ and then increases to infinity.

Now, take δ_o close to δ'_o , δ_1 as close to 1 as

$$E_{q,r}(\frac{\delta_1\lambda - 1}{\delta_1 - 1}u_{\ell_1}) < \sup_{\lambda \in (\frac{1}{\delta'_o}, \frac{1}{\delta_1})} E_{q,r}((1 - \delta_o\lambda)(\delta_1\lambda - 1)\Phi) = E_{q,r}(\frac{1}{\delta_2}\Phi)$$

and consider the curve

$$g(\lambda) = \begin{cases} (1 - \delta_o\lambda)u_{\ell_1}, & 0 \leq \lambda \leq \frac{1}{\delta_o} \\ (1 - \delta_o\lambda)(\delta_1\lambda - 1)\Phi, & \frac{1}{\delta_o} \leq \lambda \leq \frac{1}{\delta_1} \\ \frac{\delta_1\lambda - 1}{\delta_1 - 1}u_{\ell_2}, & \frac{1}{\delta_1} \leq \lambda \leq 1. \end{cases}$$

Suppose that the condition of the lemma is satisfied, then in a neighborhood $V_{(p^*, p^{**})}$ of (p^*, p^{**}) we can assume for every $(q, r) \in V_{(p^*, p^{**})}$ that

$$0 < \sup_{\lambda \in [0,1]} E_{q,r}(\lambda\Phi) < E_{q,r}(u) + \frac{p}{n} \left[\sup_M f \right]^{1-\frac{n}{p}} K_1^{-n}$$

then

$$\begin{aligned}\mu_{q,r} &\leq \sup_{\lambda \in [0,1]} E_{q,r}(g(\lambda)) \leq E_{q,r}\left(\frac{1}{4\delta_o}\Phi\right) \\ &\leq E_{q,r}(u) + \frac{p}{n} \left[\sup_M f \right]^{1-\frac{n}{p}} K_1^{-n}\end{aligned}$$

□

As a result, we obtain the following lemma

Lemma 6.4. *Suppose that the functions f and K satisfy*

$$\frac{p(n-1)}{n(p-1)} (\sup_M f)^{-\frac{n-p}{p}} (\sup_{\partial M} K)^{\frac{n-p}{p-1}} K_1^{-n} K_2^{\frac{p(n-1)}{p-1}} \leq 1$$

and that there exists a function $\Phi \in H_1^p(M)$ with $I(\Phi) > 0$, $\int_M f \Phi^{p^} dv_g > 0$, $\int_{\partial M} K \Phi^{p^{**}} d\sigma_g > 0$ such that*

$$0 < \sup_{\lambda \in [0,1]} E(\lambda \Phi) < E(u) + \frac{p}{n} \left[\sup_M f \right]^{1-\frac{n}{p}} K_1^{-n}$$

Then none of the following cases $\lim_{(q,r) \rightarrow (p^, p^{**})} v_{q,r} = 0$ or*

$$\lim_{(q,r) \rightarrow (p^*, p^{**})} \|v_{q,r} - u\|_{p,M}^p = \lim_{(q,r) \rightarrow (p^*, p^{**})} \|v_{q,r} - u\|_{p,\partial M}^p = 0$$

can occur.

Proof. If the sequence $v_{q,r}$ converges to zero function as (q,r) goes to (p^*, p^{**}) , we can repeat the proof of lemma 6.3 without the term $E(u)$ we get

$$\lim_{(q,r) \rightarrow (p^*, p^{**})} \mu_{q,r} \geq \frac{p}{n} \left[\sup_M f \right]^{1-\frac{n}{p}} K_1^{-n}$$

Since we have $E(u) < 0$, under the hypothesis of the lemma we get by lemmas 6.3 a contradiction, that is, $v_{q,r}$ does not converge to zero function as (q,r) goes to (p^*, p^{**}) and at the same time the sequence $v_{q,r}$ can not satisfy that

$$\lim_{(q,r) \rightarrow (p^*, p^{**})} \|v_{q,r} - u\|_{p,M}^p = \lim_{(q,r) \rightarrow (p^*, p^{**})} \|v_{q,r} - u\|_{p,\partial M}^p = 0$$

since $E_{q,r}(v_{q,r}) = \lim_{(q,r) \rightarrow (p^*, p^{**})} \mu_{q,r} > 0$. □

Now the proof of theorem 3.3 follows

Proof of theorem 3.3. Let u be the solution of problem (1.1) given by theorem 3.1. Let $v_{q,r}$ the sequence of solution of the subcritical problem (3.1) obtained by the mountain pass lemma. By lemma 6.1 it is bounded in $H_1^p(M)$, then after passing to a subsequence we assume that $v_{q,r}$ converges, when (q,r) goes to (p^*, p^{**}) to a function v weakly in $H_1^p(M)$, strongly in $L_p(M)$ and $L_p(\partial M)$, almost everywhere in M and in the sense of trace on ∂M and it converges to v^{p^*-1} weakly in $L_{\frac{p^*}{p^*-1}}(M)$ and to $v^{p^{**}-1}$ weakly in $L_{\frac{p^{**}}{p^{**}-1}}(\partial M)$. Then we get that v is a weak solution of problem (1.1).

Under hypothesis of the theorem we get by lemmas 6.1, 6.2, 6.3 and 6.4 that $v \neq 0$ and $v \neq u$, that is problem (1.1) admits a second weak solution. □

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